International Journal of Theoretical Physics, Vol. 45, No. 7, July 2006 (© 2006)
DOI: 10.1007/s10773-006-9150-z

The Three-Dimensional Quantum Hamilton-Jacobi Equation and Microstates

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Received November 8, 2005; accepted April 3, 2006
Published Online: June 7, 2006

In a stationary case and for any potential, we solve the three-dimensional quantum Hamilton-Jacobi equation in terms of the solutions of the corresponding Schrödinger equation. Then, in the case of separated variables, by requiring that the conjugate momentum be invariant under any linear transformation of the solutions of the Schrödinger equation used in the reduced action, we clearly identify the integration constants successively in one, two and three dimensions. In each of these cases, we analytically establish that the quantum Hamilton-Jacobi equation describes microstates not detected by the Schrödinger equation in the real wave function case.

KEY WORDS: quantum stationary Hamilton-Jacobi equation; higher dimensions; linear transformations; microstates.

PACS numbers: 03.65.Ca; 03.65.Ta; 02.30.Jr.

1. INTRODUCTION

Microstates were first introduced by Floyd (1982, 1996a,b, 2000) and investigated by other authors (Faraggi and Matone, 1998a, 2000; Carroll, 1999; Bouda, 2001). They represent physical states predicted by the quantum Hamilton-Jacobi equation but not detected by the Schrödinger wave function. Up to now, all the analytical descriptions of microstates are considered in one dimension. In this paper, one of our principal objective is their description in higher dimensions.

Recently, quantum mechanics was derived from an equivalence postulate in the one-dimensional space by Faraggi-Matone (Faraggi and Matone, 1998a,b,c, 1999, 2000; Matone, 2002a,b). These authors, together with Bertoldi, extended their finding to higher dimensions (Bertoldi *et al.*, 2000). In particular, they established a new version of the quantum stationary Hamilton-Jacobi equation (QSHJE)

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given by the two relations

$$\frac{1}{2m}(\vec{\nabla} S_0)^2 - \frac{\hbar^2}{2m}\frac{\Delta R}{R} + V(x, y, z) = E, \tag{1}$$

$$\vec{\nabla} \cdot (R^2 \vec{\nabla} S_0) = 0, \tag{2}$$

for a non-relativistic spinless particle of mass m and energy E in an external potential $V(x, y, z)$. Relations (1) and (2) represent a new version of the QSHJE because the reduced action S_0 and the function R are related to the Schrödinger wave function by

$$\Psi = R\left[\alpha\ \exp\left(i\frac{S_0}{\hbar}\right) + \beta\ \exp\left(-i\frac{S_0}{\hbar}\right)\right], \tag{3}$$

α and β being complex constants. This relation is also reproduced in Bouda (2001) where (1) and (2) are derived from the Schrödinger equation (SE) by appealing to the probability current. By setting

$$\alpha = |\alpha|\ \exp(ia), \qquad \beta = |\beta| \exp(ib), \tag{4}$$

where a and b are real parameters, expression (3) of the wave function can be written as (Bouda, 2001)

$$\Psi = R \exp\left(i\frac{a+b}{2}\right)\left[(|\alpha| + |\beta|)\cos\left(\frac{S_0}{\hbar} + \frac{a-b}{2}\right)\right.$$
$$\left. + i\,(|\alpha| - |\beta|)\sin\left(\frac{S_0}{\hbar} + \frac{a-b}{2}\right)\right]. \tag{5}$$

In Bohm's theory (Bohm, 1952a,b), in which $\alpha = 1$ and $\beta = 0$, the reduced action S_0 is a constant in the case where the wave function is real, up to a constant phase factor. However, with expression (3), S_0 is never constant. In particular, we clearly see from (5) that the reality of the wave function is expressed by $|\alpha| = |\beta|$ and not by $S_0 = cte$.

On the other hand, many suggestions to formulate the quantum trajectory equations were proposed (Bohm, 1952a; Floyd, 1982, 2000; Bouda and Djama, 2001, 2002a,b; Bouda and Hammad, 2002). In a recent paper (Bouda, 2003), the QSHJE in one dimension,

$$\frac{1}{2m}\left(\frac{\partial S_0}{\partial x}\right)^2 + V(x) - E = \frac{\hbar^2}{4m}$$
$$\times\left[\frac{3}{2}\left(\frac{\partial S_0}{\partial x}\right)^{-2}\left(\frac{\partial^2 S_0}{\partial x^2}\right)^2 - \left(\frac{\partial S_0}{\partial x}\right)^{-1}\left(\frac{\partial^3 S_0}{\partial x^3}\right)\right], \tag{6}$$

is reproduced from a general lagrangian depending on coordinate x and its higher temporal derivatives $(\dot{x}, \ddot{x}, \ldots)$ by appealing to the dimensional analysis. In particular, it is analytically established that the resulting quantum law of motion is

$$m\dot{x} = \frac{\partial S_0}{\partial x}, \tag{7}$$

recalling the Bohm relation. However, in contrast to Bohm's theory (Bohm, 1952a,b) where S_0 is deduced from the wave function (3) by setting $\alpha = 1$ and $\beta = 0$, in Bouda (2003), S_0 represents the solution of the third order differential Eq. (6). The extension of relation (7) to three dimensions can be sensibly assumed as

$$m\dot{x} = \frac{\partial S_0}{\partial x}, \qquad m\dot{y} = \frac{\partial S_0}{\partial y}, \qquad m\dot{z} = \frac{\partial S_0}{\partial z}. \tag{8}$$

Here, S_0 must be a solution of the couple of relations (1) and (2).

The paper is organized as follows. In Section 2, we solve in three dimensions the QSHJE for any potential. In Section 3, we consider the case of separated variables and identify the integration constants of the reduced action. We then investigate in Section 4 microstates in one and higher dimensions. Finally, we devote Section 5 to conclusion.

2. THE THREE-DIMENSIONAL SOLUTION OF THE QSHJE

From Eq. (5), we can deduce that (Bouda, 2001)

$$S_0 = \hbar \arctan\left(\frac{|\alpha| + |\beta|}{|\alpha| - |\beta|} \frac{\mathrm{Im}\left[\exp\left(-i(a+b)/2\right)\Psi\right]}{\mathrm{Re}\left[\exp\left(-i(a+b)/2\right)\Psi\right]} \right) + \hbar\, \frac{b-a}{2}. \tag{9}$$

The corresponding Bohm's relation can be easily obtained by taking in this last relation $|\alpha| = 1$ and $|\beta| = a = b = 0$ $(\alpha = 1, \beta = 0)$. Since the stationary SE

$$-\frac{\hbar^2}{2m}\Delta\psi + V(x, y, z)\,\psi = E\psi, \tag{10}$$

is linear, and taking into account the fact that for any solution ϕ of relation (10) then $\mathrm{Re}\,\phi$ and $\mathrm{Im}\,\phi$ are also solutions, expression (9) and its corresponding Bohm's one strongly suggest to search for the QSHJE (Eqs. (1) and (2)) a solution in the following form

$$S_0 = \hbar \arctan\left(\frac{\psi_1}{\psi_2}\right) + \hbar l, \tag{11}$$

where ψ_1 and ψ_2 are two real independent solutions of the SE, Eq. (10), and l an arbitrary dimensionless constant. Setting

$$U = \frac{\psi_1}{\psi_2}, \tag{12}$$

we have

$$\vec{\nabla} S_0 = \frac{\hbar \vec{\nabla} U}{1 + U^2}. \tag{13}$$

Substituting this expression in (2), we obtain

$$2\vec{\nabla} R \cdot \vec{\nabla} U - \frac{2UR}{1+U^2}(\vec{\nabla} U)^2 + R\Delta U = 0. \tag{14}$$

Using the fact that ψ_1 and ψ_2 solve (10), from (12) we can deduce that

$$\Delta U = -2\frac{\vec{\nabla}\psi_2}{\psi_2} \cdot \vec{\nabla} U. \tag{15}$$

It follows that relation (14) takes the form

$$\left(\frac{\vec{\nabla} R}{R} - \frac{U\vec{\nabla} U}{1+U^2} - \frac{\vec{\nabla}\psi_2}{\psi_2}\right) \cdot \vec{\nabla} U = 0. \tag{16}$$

As ψ_1 and ψ_2 are independent solutions of (10), in general $\vec{\nabla} U$ does not vanish and is not perpendicular to the vector in brackets appearing in (16). It follows that

$$\frac{\vec{\nabla} R}{R} = \frac{U\vec{\nabla} U}{1+U^2} + \frac{\vec{\nabla}\psi_2}{\psi_2}. \tag{17}$$

Substituting this relation in the identity

$$\frac{\Delta R}{R} = \vec{\nabla} \cdot \left(\frac{\vec{\nabla} R}{R}\right) + \left(\frac{\vec{\nabla} R}{R}\right)^2 \tag{18}$$

and using (15), we obtain

$$\frac{\Delta R}{R} = \left(\frac{\vec{\nabla} U}{1+U^2}\right)^2 + \frac{\Delta\psi_2}{\psi_2}. \tag{19}$$

Using (13) and taking into account the fact that ψ_2 solves (10), relation (19) becomes

$$\frac{\Delta R}{R} = \left(\frac{\vec{\nabla} S_0}{\hbar}\right)^2 + \frac{2m\,(V-E)}{\hbar^2}. \tag{20}$$

This expression is equivalent to Eq. (1). This result means that expression (11) for S_0 is a solution of the QSJHE, (Eqs. (1) and (2)).

Now, let us determine the expression of R. Substituting (12) in (17), we obtain

$$\frac{\vec{\nabla} R}{R} = \frac{1}{2} \frac{\vec{\nabla}\left(\psi_1^2 + \psi_2^2\right)}{\left(\psi_1^2 + \psi_2^2\right)}, \tag{21}$$

which leads to

$$\vec{\nabla}\left[\ln \frac{R}{\sqrt{\psi_1^2 + \psi_2^2}}\right] = \vec{0}. \tag{22}$$

Finally, we obtain

$$R = c\sqrt{\psi_1^2 + \psi_2^2}, \tag{23}$$

where c is an integration constant.

Of course, a direct substitution of expressions (11) and (23) in (1) and (2) allows, with the use of the SE, to check that these expressions are indeed solutions of (1) and (2).

3. THE CASE OF SEPARATED VARIABLES

3.1. The One-Dimensional Case

Before we examine the higher-dimensional cases, it is instructive to consider the problem of identifying the integration constants in one dimension. The SE, Eq. (10), reduces to

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\,\psi = E\psi. \tag{24}$$

Let (ϕ_1, ϕ_2) be a set of two real independent solutions of (24). Since the SE is linear and always admits two real independent solutions, in order to make visible all the integration constants in (11), let us write the real functions ψ_1 and ψ_2 in the general form

$$\psi_1 = \nu_1\phi_1 + \nu_2\phi_2, \qquad \psi_2 = \mu_1\phi_1 + \mu_2\phi_2, \tag{25}$$

where $(\nu_1, \nu_2, \mu_1, \mu_2)$ are arbitrary real constants satisfying the condition $\nu_1\mu_2 \neq \nu_2\mu_1$ which guarantees that ψ_1 and ψ_2 are independent. Relation (11) turns out to be

$$S_0 = \hbar\ \arctan\left(\frac{\nu_1\phi_1 + \nu_2\phi_2}{\mu_1\phi_1 + \mu_2\phi_2}\right) + \hbar l. \tag{26}$$

This expression must be a general solution of the one-dimensional QSHJE, Eq. (6). As explained in Bouda (2001), Eq. (6) is a second order differential equation with respect to $\partial S_0/\partial x$, then this derivative must depend only on two integration constants. Therefore, the function S_0 contains a further constant which must be additive. It is represented by $\hbar l$ in (26). Thus, we can set $\nu_1 = \mu_2 = 1$ and interpret $(\mu_1, \nu_2, \hbar l)$ as integration constants of S_0.

However, the extension of this reasoning to higher dimensions is not trivial even in the case of separated variables. In fact, in two or three dimensions we have not an ordinary differential equation, but a couple of two partial differential equations, Eqs. (1) and (2). For example in three dimensions, where the potential takes the following form $V(x, y, z) = V_x(x) + V_y(y) + V_z(z)$, if we tempt to search for Eqs. (1) and (2) solutions with the standard method by writing S_0 in the form

$$S_0(x, y, z) = S_{0x}(x) + S_{0y}(y) + S_{0z}(z),$$

and use for R a form as the one given in Bertoldi *et al.* (2000)

$$R(x, y, z) = R_x(x)R_y(y)R_z(z),$$

the three separated equations which result from Eqs. (1) and (2) differ from the usual one-dimensional QSHJE and lead to a deadlock. This is the reason for which we will resolve this problem with a novel approach. We first review the problem in one dimension and reproduce the expected results. This approach consists in determining the minimum number of parameters in the set $(\nu_1, \nu_2, \mu_1, \mu_2)$, which we must keep free in the expression of S_0, but sufficient to guarantee the invariance of the conjugate momentum

$$\frac{\partial S_0}{\partial x} = \frac{\partial \tilde{S}_0}{\partial x} \tag{27}$$

under an arbitrary linear transformation of the couple (ϕ_1, ϕ_2)

$$\phi_i \to \theta_i = \sum_{j=1}^{2} \alpha_{ij}\phi_j, \qquad i = 1, 2 \tag{28}$$

$\tilde{S}_0$ being the new reduced action and α_{ij} arbitrary real constant parameters. In other words, if we choose another couple (θ_1, θ_2) of the solutions of SE instead of (ϕ_1, ϕ_2) and write the reduced action in the same form as in (26),

$$\tilde{S}_0 = \hbar \ \arctan\left(\frac{\tilde{\nu}_1\theta_1 + \tilde{\nu}_2\theta_2}{\tilde{\mu}_1\theta_1 + \tilde{\mu}_2\theta_2}\right) + \hbar\tilde{l}, \tag{29}$$

the equation of motion, as relation (7), must remain unchanged, meaning that our mathematical choice does not affect the physical result. In this procedure, we have to accomplish two principal simultaneous tasks. The first is to prove for any transformation (28) the existence of the parameters $(\tilde{\nu}_1, \tilde{\nu}_2, \tilde{\mu}_1, \tilde{\mu}_2)$, used in

$\tilde{S}_0$ instead of $(\nu_1, \nu_2, \mu_1, \mu_2)$, in such a way as to guarantee the invariance (27). The second task consists in eliminating the maximum of parameters in the set $(\nu_1, \nu_2, \mu_1, \mu_2)$ as in $(\tilde{\nu}_1, \tilde{\nu}_2, \tilde{\mu}_1, \tilde{\mu}_2)$ without violating relation (27) and without inducing any restriction on transformation (28). We would like to add that this idea of invariance was first introduced in Bouda and Djama (2002b). However the goal in this reference was not to determine the minimum number of pertinent parameters which would be playing the role of integration constants, but it was only to check that the choice of the couple of solutions (ϕ_1, ϕ_2) does not affect the equation of motion.

Before going further, it is crucial to note that the extension to higher dimensions of the invariance of the derivatives of S_0 will induce insurmountable calculations. This is the reason for which we turn condition (27) into the invariance of S_0,

$$S_0 = \tilde{S}_0 + \hbar l_0, \tag{30}$$

up to an additive constant $\hbar l_o$. It is then interesting to remark that for two arbitrary functions f and g, if

$$\arctan(f) = \arctan(g) + l_1, \tag{31}$$

we can easily deduce that

$$f - g = k_1 (1 + fg), \tag{32}$$

where $k_1 = \tan(l_1)$. Then, substituting (26) and (29) in (30) and applying (31) and (32), we deduce

$$\sum_{i=1}^{2} \sum_{j=1}^{2} [k(\mu_i \tilde{\mu}_j + \nu_i \tilde{\nu}_j) + \tilde{\nu}_j \mu_i - \nu_i \tilde{\mu}_j] \phi_i \theta_j = 0, \tag{33}$$

where

$$k = \tan\left(l_0 + \tilde{l} - l\right). \tag{34}$$

From transformation (28), we can deduce ϕ_i

$$\phi_i = \sum_{l=1}^{2} \beta_{il}\, \theta_l, \tag{35}$$

where β_{ij} can be determined by the system of four equations

$$\sum_{j=1}^{2} \beta_{ij} \alpha_{jl} = \delta_{il}. \tag{36}$$

Substituting (35) in (33), we find

$$\sum_{j=1}^{2}\sum_{l=1}^{2}\left[\tilde{\mu}_j \sum_{i=1}^{2}(k\mu_i - \nu_i)\,\beta_{il} + \tilde{\nu}_j \sum_{i=1}^{2}(k\nu_i + \mu_i)\,\beta_{il}\right]\theta_l\theta_j = 0. \tag{37}$$

This equation contains three independent terms and then the coefficients which precede θ_1^2, θ_2^2 and $\theta_1\theta_2$ must take a vanishing value

$$\tilde{\mu}_1 \sum_{i=1}^{2}(k\mu_i - \nu_i)\,\beta_{i1} + \tilde{\nu}_1 \sum_{i=1}^{2}(k\nu_i + \mu_i)\,\beta_{i1} = 0, \tag{38}$$

$$\tilde{\mu}_2 \sum_{i=1}^{2}(k\mu_i - \nu_i)\,\beta_{i2} + \tilde{\nu}_2 \sum_{i=1}^{2}(k\nu_i + \mu_i)\,\beta_{i2} = 0, \tag{39}$$

$$\tilde{\mu}_1 \sum_{i=1}^{2}(k\mu_i - \nu_i)\,\beta_{i2} + \tilde{\nu}_1 \sum_{i=1}^{2}(k\nu_i + \mu_i)\,\beta_{i2} + \tilde{\mu}_2 \sum_{i=1}^{2}(k\mu_i - \nu_i)\,\beta_{i1} + \tilde{\nu}_2 \sum_{i=1}^{2}(k\nu_i + \mu_i)\,\beta_{i1} = 0. \tag{40}$$

Thus, we have three independent equations and four unknown parameters: $\tilde{\nu}_1$, $\tilde{\nu}_2$, $\tilde{\mu}_1$ and $\tilde{\mu}_2$. However, if we divide by $\tilde{\nu}_1$ in the quotient appearing in (29) and define $\tilde{\nu}_2/\tilde{\nu}_1$, $\tilde{\mu}_1/\tilde{\nu}_1$ and $\tilde{\mu}_2/\tilde{\nu}_1$ as new parameters, it amounts to setting $\tilde{\nu}_1 = 1$ and keeping $\tilde{\nu}_2$, $\tilde{\mu}_1$ and $\tilde{\mu}_2$ unchanged. Of course, we must also take $\nu_1 = 1$ since the same form for S_0 and $\tilde{S}_0$ is required. Furthermore, the parameter k defined by (34) is free. A judicious choice of k allows to fix one of the parameters $\tilde{\nu}_2$, $\tilde{\mu}_1$ and $\tilde{\mu}_2$. For example, if we take

$$k = \frac{\sum_{i=1}^{2}(\nu_i\beta_{i1} - \mu_i\beta_{i2})}{\sum_{i=1}^{2}(\nu_i\beta_{i2} + \mu_i\beta_{i1})}, \tag{41}$$

we can check that the system (38), (39) and (40) gives $\tilde{\mu}_2 = 1$ and allows to express $\tilde{\nu}_2$ and $\tilde{\mu}_1$ in terms of μ_i, ν_i and β_{ij}. Of course, for the same reason as above, we must also take $\mu_2 = 1$. As β_{ij} can be expressed in terms of α_{ij} from (36), if we add any condition on $\tilde{\nu}_2$ or $\tilde{\mu}_1$, the system (38), (39) and (40) will induce a relation between μ_1, ν_2 and α_{ij}. Since μ_1 and ν_2 correspond to the initial choice (ϕ_1, ϕ_2) used in the initial reduced action, this relation between μ_1, ν_2 and α_{ij} will represent a restriction on transformation (28) and then on the choice of the couple (θ_1, θ_2). Thus, if we want to guarantee the invariance of $\partial S_0/\partial x$ under any linear transformation, we must keep free (ν_2, μ_1) in the expression of S_0, as we must do it for $(\tilde{\nu}_2, \tilde{\mu}_1)$ if we choose to deal with $\tilde{S}_0$. Finally, we reach the same conclusion as at the beginning of this section where these two pertinent parameters (ν_2, μ_1) are identified as integration constants of the reduced action S_0, expression (26).

3.2. The Two-Dimensional Case

Let us consider the two-dimensional case in which the potential takes the form

$$V(x, y) = V_x(x) + V_y(y). \tag{42}$$

Writing $\psi(x, y) = X(x)Y(y)$, the two-dimensional SE leads to

$$-\frac{\hbar^2}{2m}\frac{d^2X}{dx^2} + V_xX = E_xX, \tag{43}$$

$$-\frac{\hbar^2}{2m}\frac{d^2Y}{dy^2} + V_yY = E_yY, \tag{44}$$

where E_x and E_y are real constants satisfying

$$E_x + E_y = E. \tag{45}$$

Let us call (X_1, X_2) and (Y_1, Y_2) two couples of real independent solutions respectively of (43) and (44). Then, the two-dimensional SE admits four independent solutions

$$\phi_1 = X_1Y_1, \quad \phi_2 = X_1Y_2, \quad \phi_3 = X_2Y_1, \quad \phi_4 = X_2Y_2. \tag{46}$$

As in one dimension, the general form of the reduced action is obtained from (11) by writing

$$S_0 = \hbar\ \arctan\left(\frac{\sum_{i=1}^4 \nu_i\phi_i}{\sum_{i=1}^4 \mu_i\phi_i}\right) + \hbar l, \tag{47}$$

$(\nu_1, \ldots, \nu_4, \mu_1, \ldots, \mu_4)$ being arbitrary real constants satisfying a condition with which ψ_1 and ψ_2 are not proportional. The equations of motion, as relations (8), are now obtained from $\partial S_0/\partial x$ and $\partial S_0/\partial y$. Then, let us impose the invariance of the conjugate momentum components

$$\frac{\partial S_0}{\partial x} = \frac{\partial \tilde{S}_0}{\partial x}, \qquad \frac{\partial S_0}{\partial y} = \frac{\partial \tilde{S}_0}{\partial y} \tag{48}$$

under the following arbitrary linear transformation

$$\phi_i \to \theta_i = \sum_{j=1}^4 \alpha_{ij}\phi_j, \qquad i = 1, 2, 3, 4 \tag{49}$$

where $\tilde{S}_0$ is the new reduced action defined as in (47)

$$\tilde{S}_0 = \hbar \ \arctan\left(\frac{\sum_{i=1}^{4} \tilde{\nu}_i \theta_i}{\sum_{i=1}^{4} \tilde{\mu}_i \theta_i}\right) + \hbar \tilde{l}, \tag{50}$$

and α_{ij} are arbitrary real constant parameters. It is easy to show that conditions (48) can be turned into $S_0 = \tilde{S}_0 + \hbar l_0$ as in (30). Thus, with the use of (31) and (32), we obtain

$$\sum_{i=1}^{4}\sum_{j=1}^{4}[k(\mu_i \tilde{\mu}_j + \nu_i \tilde{\nu}_j) + \mu_i \tilde{\nu}_j - \nu_i \tilde{\mu}_j]\phi_i \theta_j = 0, \tag{51}$$

where k is given as in (34). In contrast to the one-dimensional case and for reasons that will be clarified farther, we will make from (51) an expansion in $\phi_i \phi_l$ and not in $\theta_i \theta_l$. Thus, substituting (49) in (51), we get to

$$\sum_{i=1}^{4}\sum_{l=1}^{4}\left[\nu_i \sum_{j=1}^{4}(k\tilde{\nu}_j - \tilde{\mu}_j)\alpha_{jl} + \mu_i \sum_{j=1}^{4}\left(k\tilde{\mu}_j + \tilde{\nu}_j\right)\alpha_{jl}\right]\phi_i \phi_l = 0. \tag{52}$$

Setting

$$A_l = \sum_{j=1}^{4}(k\tilde{\nu}_j - \tilde{\mu}_j)\alpha_{jl}, \quad B_l = \sum_{j=1}^{4}(k\tilde{\mu}_j + \tilde{\nu}_j)\alpha_{jl}, \tag{53}$$

Eq. (52) becomes

$$\sum_{i=1}^{4}\sum_{l=1}^{4}(A_l\, \nu_i + B_l\, \mu_i)\, \phi_i \phi_l = 0. \tag{54}$$

In this equation, we have sixteen terms ($4 \times 4 = 16$). The symmetry $\phi_i \phi_l = \phi_l \phi_i$ reduces the number of terms to ten [$(16 - 4)/2 + 4 = 10$]. From (46), we have also $\phi_1 \phi_4 = \phi_2 \phi_3$. Thus, in (54) we have nine independent terms. For $(i, l) \notin \{(1, 4), (4, 1), (2, 3), (3, 2)\}$, we have eight equations

$$A_l\, \nu_i + B_l\, \mu_i + A_i\, \nu_l + B_i\, \mu_l = 0. \tag{55}$$

In the case where $i = l$, (55) gives four equations

$$\mu_i = -\frac{A_i}{B_i}\nu_i, \qquad i = 1, 2, 3, 4 \tag{56}$$

and then, by taking into account this result, (55) gives the other four equations for $i \neq l$

$$\frac{\nu_i}{B_i} = \frac{\nu_l}{B_l}, \tag{57}$$

meaning that $\nu_1/B_1 = \nu_2/B_2 = \nu_3/B_3 = \nu_4/B_4$. As $\phi_1\phi_4 = \phi_2\phi_3$, the ninth equation is obtained by combining the cases $(i, l) = (1, 4)$ and $(i, l) = (2, 3)$

$$A_4\,\nu_1 + B_4\,\mu_1 + A_1\,\nu_4 + B_1\,\mu_4 + A_3\,\nu_2 + B_3\,\mu_2 + A_2\,\nu_3 + B_2\,\mu_3 = 0. \quad (58)$$

With the use of (56) and (57), it is easy to check that (58) represents an identity. In (56) we have four independent equations but in (57) only three. Thus, we have seven linear equations and eight unknown parameters: $(\tilde{\nu}_1, \ldots, \tilde{\nu}_4, \tilde{\mu}_1, \ldots, \tilde{\mu}_4)$ which are present through A_i and B_i. However, as in one dimension, the operation consisting in dividing by $\tilde{\nu}_1$ in the quotient appearing in (50) amounts to setting $\tilde{\nu}_1 = 1$ in (50) and $\nu_1 = 1$ in (47). Now, we have seven equations with seven unknown parameters. In addition, k being free, we can choose its value in order to obtain for example $\tilde{\mu}_4 = 1$, and then to also take $\mu_4 = 1$. This amounts to solving our system of seven equations with respect to the following seven parameters: $(\tilde{\nu}_2, \tilde{\nu}_3, \tilde{\nu}_4, \tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3, k)$. As in one dimension, any further condition on $(\tilde{\nu}_2, \tilde{\nu}_3, \tilde{\nu}_4, \tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3)$ will induce restrictions on the linear transformation (49). In conclusion, the number of pertinent parameters of the reduced action (47), with which we can reproduce the same equations of motion under any linear transformation, is six. As we have fixed $\nu_1 = \mu_4 = 1$, the pertinent parameters playing the role of integration constants are $(\nu_2, \nu_3, \nu_4, \mu_1, \mu_2, \mu_3)$.

We would like to add that, in contrast to the one-dimensional case, we have made in (52) an expansion in $\phi_i\phi_l$ but not in $\theta_i\theta_l$. The reason is that in two dimensions, the ten product $\phi_i\phi_l$ are not linearly independent since we have seen that $\phi_1\phi_4 = \phi_2\phi_3$. This implies a more complicated relation between the ten products $\theta_i\theta_l$ and will induce a tedious calculations if we make the expansion in $\theta_i\theta_l$.

3.3. The Three-Dimensional Case

In the same manner, let us now consider the three-dimensional case and write the potential in the following form

$$V(x, y, z) = V_x(x) + V_y(y) + V_z(z). \quad (59)$$

Writing $\psi(x, y, z) = X(x)Y(y)Z(z)$, the SE in three dimensions, Eq. (10), leads to

$$-\frac{\hbar^2}{2m}\frac{d^2X}{dx^2} + V_x X = E_x X, \quad (60)$$

$$-\frac{\hbar^2}{2m}\frac{d^2Y}{dy^2} + V_y Y = E_y Y, \quad (61)$$

$$-\frac{\hbar^2}{2m}\frac{d^2Z}{dz^2} + V_z Z = E_z Z, \quad (62)$$

where E_x, E_y and E_z are real constants satisfying

$$E_x + E_y + E_z = E. \tag{63}$$

Let us call (X_1, X_2), (Y_1, Y_2) and (Z_1, Z_2) three couples of real independent solutions respectively of (60), (61) and (62). It follows that the SE in three dimensions admits eight real independent solutions

$$\begin{cases} \phi_1 = X_1Y_1Z_1,\ \phi_2 = X_1Y_1Z_2,\ \phi_3 = X_1Y_2Z_1,\ \phi_4 = X_1Y_2Z_2, \\ \phi_5 = X_2Y_1Z_1,\ \phi_6 = X_2Y_1Z_2,\ \phi_7 = X_2Y_2Z_1,\ \phi_8 = X_2Y_2Z_2. \end{cases} \tag{64}$$

The general form of the reduced action can be deduced from (11) by writing

$$S_0 = \hbar\ \arctan\left(\frac{\sum_{i=1}^{8} \nu_i \phi_i}{\sum_{i=1}^{8} \mu_i \phi_i}\right) + \hbar l. \tag{65}$$

As in the previous cases, in order to determine the pertinent parameters among the real constants $(\nu_1, \ldots, \nu_8, \mu_1, \ldots, \mu_8)$, let us impose the invariance of the conjugate momentum components

$$\frac{\partial S_0}{\partial x} = \frac{\partial \tilde{S}_0}{\partial x}, \qquad \frac{\partial S_0}{\partial y} = \frac{\partial \tilde{S}_0}{\partial y}, \qquad \frac{\partial S_0}{\partial z} = \frac{\partial \tilde{S}_0}{\partial z} \tag{66}$$

under the following arbitrary linear transformation

$$\phi_i \to \theta_i = \sum_{j=1}^{8} \alpha_{ij}\phi_j, \qquad i = 1, \ldots, 8 \tag{67}$$

where $\tilde{S}_0$ is the new reduced action defined as in (65)

$$\tilde{S}_0 = \hbar\ \arctan\left(\frac{\sum_{i=1}^{8} \tilde{\nu}_i \theta_i}{\sum_{i=1}^{8} \tilde{\mu}_i \theta_i}\right) + \hbar \tilde{l}, \tag{68}$$

and α_{ij} are arbitrary real constant parameters. Conditions (66) can be also turned into $S_0 = \tilde{S}_0 + \hbar l_0$. Thus, as in two dimensions, we deduce that

$$\sum_{i=1}^{8}\sum_{l=1}^{8}(A_l\, \nu_i + B_l\, \mu_i)\phi_i\phi_l = 0, \tag{69}$$

where

$$A_l = \sum_{j=1}^{8}(k\tilde{\nu}_j - \tilde{\mu}_j)\alpha_{jl},\ B_l = \sum_{j=1}^{8}(k\tilde{\mu}_j + \tilde{\nu}_j)\alpha_{jl}. \tag{70}$$

In (69), we have sixty-four terms $(8 \times 8 = 64)$. The symmetry $\phi_i\phi_l = \phi_l\phi_i$ reduces this number to thirty-six $[(64 - 8)/2 + 8 = 36]$. From (64), all possible relations which we can deduce between the products $\phi_i\phi_l$ are

$$\begin{cases} \phi_1\phi_4 = \phi_2\phi_3,\ \phi_1\phi_6 = \phi_2\phi_5,\ \phi_1\phi_7 = \phi_3\phi_5, \\ \phi_2\phi_8 = \phi_4\phi_6,\ \phi_3\phi_8 = \phi_4\phi_7,\ \phi_5\phi_8 = \phi_6\phi_7 \end{cases} \tag{71}$$

and

$$\phi_1\phi_8 = \phi_2\phi_7 = \phi_3\phi_6 = \phi_4\phi_5. \tag{72}$$

In (71) and (72) we have nine independent relations. It follows that in (69), we have twenty-seven independent terms ($36 - 9 = 27$). For the terms $i = l$, we deduce eight relations

$$A_i\, \nu_i + B_i\, \mu_i = 0, \tag{73}$$

leading to

$$\mu_i = -\frac{A_i}{B_i}\nu_i, \qquad i = 1, \ldots, 8. \tag{74}$$

If

$$(i, l) \in \{(1,2),\ (1,3),\ (1,5),\ (2,4),\ (2,6),\ (3,4),\ (3,7),\ (4,8),\ (5,6),\ (5,7),\ (6,8),\ (7,8)\}, \tag{75}$$

from (69) we deduce the following twelve relations

$$A_l\, \nu_i + B_l\, \mu_i + A_i\, \nu_l + B_i\, \mu_l = 0. \tag{76}$$

Taking into account relations (74), these last relations lead to

$$\frac{\nu_i}{B_i} = \frac{\nu_l}{B_l}. \tag{77}$$

If we look more closely at the set given in (75), we deduce that (77) is valid $\forall\, i \in [1, 2, \ldots, 8]$ and $\forall\, l \in [1, 2, \ldots, 8]$. By using the couples of indexes appearing in (71) and (72), we can deduce the seven remaining relations ($27 - 8 - 12 = 7$). We can check that they all represent identities. For example, with the first relation in (71), we deduce from (69)

$$A_4\, \nu_1 + B_4\, \mu_1 + A_1\, \nu_4 + B_1\, \mu_4 + A_3\, \nu_2 + B_3\, \mu_2 + A_2\, \nu_3 + B_2 \mu_3 = 0. \tag{78}$$

If we substitute in this relation μ_1, μ_2, μ_3 and μ_4 by their values given in (74) and take into account (77), we easily obtain an identity. We emphasize that with (72), we have only one relation which is also an identity.

The twelve relations (76) are reduced, with the use of (74), to seven independent equations given in (77). The five missing relations are identities. This means that the system (74) and (76) is turned into the system (74) and (77) which contains fifteen ($8 + 7 = 15$) independent linear equations and sixteen unknown parameters $(\tilde{\nu}_1, \ldots, \tilde{\nu}_8, \tilde{\mu}_1, \ldots, \tilde{\mu}_8)$ which are present through A_i and B_i. However, as in the previous cases, the operation consisting in dividing by $\tilde{\nu}_1$ in the quotient appearing in (68) amounts to setting $\tilde{\nu}_1 = 1$ in (68) and then $\nu_1 = 1$ in (65), since the same form for S_0 and $\tilde{S}_0$ is required. Again, the freedom in choosing k allows us to fix another parameter. Then, if we choose for example $\tilde{\mu}_8 = 1$, we have also to take $\mu_8 = 1$. Of course, any further condition on $\tilde{\nu}_i$ or $\tilde{\mu}_i$ will induce restriction on the linear transformation (67). In conclusion, the number of pertinent parameters in (65) is fourteen, and with the above choice, these parameters are $(\nu_2, \ldots, \nu_8, \mu_1, \ldots, \mu_7)$. They play a role of integration constants of the reduced action, expression (65). In addition, we notice that the same reasons as in two dimensions have not allowed us to make in (69) an expansion in $\theta_i \theta_l$.

4. MICROSTATES

In this section, we will examine if the initial conditions, which determine completely the Schrödinger wave function, are sufficient to determine all the pertinent parameters of the reduced action. In other words, knowing that the reduced action is the generator of motion, one may wonder if, to these initial conditions of the wave function, correspond one or many trajectories of the particle.

For this purpose, remark that the constant c appearing in (23) can be absorbed in the parameters α and β when we use relation (3) or (5). Thus, in any dimension, let us write

$$R = \sqrt{\psi_1^2 + \psi_2^2} \tag{79}$$

and substitute in (5) the reduced action S_0 by its expression (11)

$$\Psi = R\left\{ (|\alpha| + |\beta|) \cos\left[\arctan\left(\frac{\psi_1}{\psi_2}\right) + l + \frac{a-b}{2}\right] + i\,(|\alpha| - |\beta|) \sin\left[\arctan\left(\frac{\psi_1}{\psi_2}\right) + l + \frac{a-b}{2}\right]\right\}, \tag{80}$$

where we have discarded the unimportant constant phase factor $\exp[i(a+b)/2]$. Since the additive constant l appearing in (11) has no dynamical effect, we can choose it equal to $(b-a)/2$. Therefore, with the use of (79) and some trigonometric relations, we have

$$R\,\cos\left[\arctan\left(\frac{\psi_1}{\psi_2}\right)\right] = \psi_2, \; R\,\sin\left[\arctan\left(\frac{\psi_1}{\psi_2}\right)\right] = \psi_1 \tag{81}$$

and Eq. (80) turns out to be

$$\Psi = (|\alpha| + |\beta|)\,\psi_2 + i\,(|\alpha| - |\beta|)\,\psi_1. \tag{82}$$

This relation is valid in any dimension. First, it allows to reproduce the well-known results in one dimension obtained in (Bouda, 2001). In fact, comparing (11) and (26), we have

$$\psi_1 = \phi_1 + \nu_2\phi_2, \qquad \psi_2 = \mu_1\phi_1 + \phi_2, \tag{83}$$

where we have set $\nu_1 = \mu_2 = 1$ as indicated in the previous section. Substituting (83) in (82), we obtain

$$\Psi = [\mu_1 (|\alpha| + |\beta|) + i (|\alpha| - |\beta|)]\, \phi_1 + [(|\alpha| + |\beta|) + i \nu_2 (|\alpha| - |\beta|)]\, \phi_2. \tag{84}$$

On the other hand, the wave function can be written as a linear combination of the two real independent solutions ϕ_1 and ϕ_2

$$\Psi = c_1\, \phi_1 + c_2\, \phi_2, \tag{85}$$

where c_1 and c_2 are generally complex constants determined by the initial or boundary conditions of the wave function. Identification of (84) and (85) leads to

$$c_1 = \mu_1 (|\alpha| + |\beta|) + i (|\alpha| - |\beta|), \tag{86}$$

$$c_2 = |\alpha| + |\beta| + i \nu_2 (|\alpha| - |\beta|). \tag{87}$$

In the case where $|\alpha| \neq |\beta|$, separating the real part from the imaginary one in (86) and (87), we obtain a system of four equations which can be solved with respect to $|\alpha|$, $|\beta|$, μ_1 and ν_2. It follows that in the complex wave function case ($|\alpha| \neq |\beta|$), the initial conditions of the wave function Ψ fix univocally the reduced action. There is no trace of microstates. In the real wave function case ($|\alpha| = |\beta|$), up to a constant phase factor, (86) and (87) do not allow to determine ν_2. Thus, for a given physical state Ψ, we have a family of trajectories, specified by the different values of ν_2, corresponding to microstates not detected by the SE. The same conclusion is also reached in Floyd (1996a). We would like to add that if we use the Bohm ansatz ($\alpha = 1, \beta = 0$), Eqs. (86) and (87) imply that $\mathrm{Im}\, c_1 = 1$ and $\mathrm{Re}\, c_2 = 1$. This is an unsatisfactory result since c_1 and c_2 are fixed by the initial conditions of the wave function and, then, we must not obtain fixed values for $\mathrm{Im}\, c_1$ and $\mathrm{Re}\, c_2$. This is also the proof that the presence of α and β in the relation between the reduced action and the Schrödinger wave function, Eq. (3), is necessary. However, among the four real parameters which define the complex number α and β, there are only two which are linked to the initial conditions of the wave function. The two others are superfluous. This has been seen in Bouda (2001) by showing that the functions R and S_0 are invariant under a dilatation and a rotation in the complex space of expression (3) of the wave function. This invariance allowed to make a transformation which fixed the two superfluous degrees of freedom. In our above reasoning we have eliminated these superfluous parameters first by discarding in (80) the phase factor $\exp{[i(a+b)/2]}$ and second by choosing $(b-a)/2$ equal to the additive integration constant l in (82). This means that we have fixed the phases a and b of α and β and kept free $|\alpha|$ and $|\beta|$. These modulus are determined by the four real equations which can be deduced from (86) and (87), meaning that $|\alpha|$ and $|\beta|$ are linked to the initial conditions of the wave function.

The two-dimensional case is similar to the three-dimensional one. For this reason, we straightforwardly investigate microstates in three dimensions. Comparing (11) and (65), we have

$$\psi_1 = \phi_1 + \sum_{i=2}^{8} \nu_i \, \phi_i, \; \psi_2 = \sum_{i=1}^{7} \mu_i \, \phi_i + \phi_8, \tag{88}$$

where we have set $\nu_1 = \mu_8 = 1$ as indicated in Section 3. The functions ϕ_i are defined in (64). Substituting (88) in (82), we obtain

$$\begin{aligned}\Psi &= [\mu_1 (|\alpha| + |\beta|) + i (|\alpha| - |\beta|)] \phi_1 \\ &\quad + \sum_{i=2}^{7} [\mu_i (|\alpha| + |\beta|) + i \nu_i (|\alpha| - |\beta|)] \phi_i \\ &\quad + [|\alpha| + |\beta| + i \nu_8 (|\alpha| - |\beta|)] \phi_8.\end{aligned} \tag{89}$$

As above, the wave function can be written as a linear combination of ϕ_i $(i = 1, \ldots, 8)$

$$\Psi = \sum_{i=1}^{8} c_i \phi_i, \tag{90}$$

where c_i are complex constants which can be determined by the boundary conditions of the wave function. Identification of (89) and (90) leads to

$$c_1 = \mu_1 (|\alpha| + |\beta|) + i (|\alpha| - |\beta|), \tag{91}$$

$$c_i = \mu_i (|\alpha| + |\beta|) + i \nu_i (|\alpha| - |\beta|), \qquad i = 2, 3, \ldots, 7 \tag{92}$$

$$c_8 = |\alpha| + |\beta| + i \nu_8 \; (|\alpha| - |\beta|). \tag{93}$$

In the complex wave function case $(|\alpha| \neq |\beta|)$, separating the real part from the imaginary one, we obtain from these relations a system of sixteen equations which can be solved with respect to the sixteen following unknown $(|\alpha|, |\beta|, \nu_2, \ldots, \nu_8, \mu_1, \ldots, \mu_7)$. As in one dimension, the knowledge of $(c_1, \ldots, c_8)$ is sufficient to fix univocally the reduced action. There is no trace of microstates. In the real wave function case $(|\alpha| = |\beta|)$, up to a constant phase factor, it is clear that the system (91), (92) and (93) does not allow to determine $(\nu_2, \nu_3, \ldots, \nu_8)$. As in one dimension, for a given physical state Ψ, we have a family of trajectories, specified by the values of $(\nu_2, \nu_3, \ldots, \nu_8)$, corresponding to microstates not detected by the SE. We also reach the same conclusion in two-dimensions, namely microstates appear only in the real wave function case. We would like to add that also in higher dimensions, $|\alpha|$ and $|\beta|$ are linked to the initial conditions of the Schrödinger wave function since these modulus are fixed by the system (91), (92) and (93), or its analogous in two dimensions. As in one dimension, the phases a and b are superfluous and have been eliminated in (82).

5. CONCLUSION

This work can be summarized in three main results.

The first concerns the resolution in three dimensions of the QSHJE represented by the couple of relations (1) and (2). For any external potential, the expressions of the couple of functions (R, S_0) are given in terms of a couple of real independent solutions of the SE. In other words, we reduced a problem of two non-linear partial differential equations to one linear equation.

The second result concerns the identification in the case of separated variables of the pertinent parameters playing a role of integration constants of the reduced action. Of course, in one dimension, the solution of this problem is already known (Floyd, 1986; Faraggi and Matone, 1998b, 1999, 2000; Bouda, 2001). However, in higher dimensions, we had to solve two coupled partial differential equations and, then, we had no means to fix the number of integration constants since the standard method does not work. We surmounted this difficulty by imposing the invariance of the reduced action, up to an additive constant, under an arbitrary linear transformation of the set of solutions of the SE. This amounts to requiring that, for any choice of the set of solutions of the SE, we reproduce the same equations of motion. In this procedure, our task consisted in determining the minimum number of parameters which we must keep free in such a way as to impose this invariance of the reduced action. We first applied this new procedure in one dimension and reproduced the expected results (Bouda, 2001). We then extended the approach to two and three dimensions.

The third principal result we obtained concerns microstates. We showed that in one dimension, as in higher dimensions, microstates appear only in the case where the Schrödinger wave function is real, up to a constant phase factor. In higher dimensions, it is the first time that microstates are analytically described. As indicated in Landau *et al.* (1967), in the case where there is no degeneracy, bound states are described by real wave functions. Thus, bound states reveal microstates not detected by the Schrödinger wave function. As concluded in the one-dimensional case (Floyd, 1996a), in higher dimensions the Schrödinger wave function does not describe exhaustively quantum phenomena. The QSHJE is more fundamental.

ACKNOWLEDGMENT

The authors would like to thank E. R Floyd for valuable discussions and suggestions.

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